19.8 CASE STUDY continued

Note that we have displayed the negative of the resultants, in order that they point "downhill."

The result is shown in Fig. 19.10. The function's peak occurs at $x = -1$ and $y = 1.5$ and then drops away in all directions. As indicated by the lengthening arrows, the gradient drops off more steeply to the northeast and the southwest.

PROBLEMS

19.1 Compute forward and backward difference approximations of $O(h)$ and $O(h^2)$, and central difference approximations of $O(h^2)$ and $O(h^4)$ for the first derivative of $y = \cos x$ at $x = \pi/4$ using a value of $h = \pi/12$. Estimate the true percent relative error ε_t for each approximation.

19.2 Use centered difference approximations to estimate the first and second derivatives of $y = e^x$ at $x = 2$ for $h = 0.1$. Employ both $O(h^2)$ and $O(h^4)$ formulas for your estimates.

19.3 Use a Taylor series expansion to derive a centered finite-difference approximation to the third derivative that is second-order accurate. To do this, you will have to use four different expansions for the points x_{i-2}, x_{i-1}, x_{i+1}, and x_{i+2}. In each case, the expansion will be around the point x_i. The interval Δx will be used in each case of $i-1$ and $i+1$, and $2\Delta x$ will be used in each case of $i-2$ and $i+2$. The four equations must then be combined in a way to eliminate the first and second derivatives. Carry enough terms along in each expansion to evaluate the first term that will be truncated to determine the order of the approximation.

19.4 Use Richardson extrapolation to estimate the first derivative of $y = \cos x$ at $x = \pi/4$ using step sizes of $h_1 = \pi/3$ and $h_2 = \pi/6$. Employ centered differences of $O(h^2)$ for the initial estimates.

19.5 Repeat Prob. 19.4, but for the first derivative of $\ln x$ at $x = 5$ using $h_1 = 2$ and $h_2 = 1$.

19.6 Employ Eq. (19.21) to determine the first derivative of $y = 2x^4 - 6x^3 - 12x - 8$ at $x = 0$ based on values at $x_0 = -0.5$, $x_1 = 1$, and $x_2 = 2$. Compare this result with the true value and with an estimate obtained using a centered difference approximation based on $h = 1$.

19.7 Prove that for equispaced data points, Eq. (19.21) reduces to Eq. (4.25) at $x = x_1$.

19.8 Develop an M-file to apply a Romberg algorithm to estimate the derivative of a given function.

19.9 Develop an M-file to obtain first-derivative estimates for unequally spaced data. Test it with the following data:

x	0.6	1.5	1.6	2.5	3.5
$f(x)$	0.9036	0.3734	0.3261	0.08422	0.01596

where $f(x) = 5e^{-2x}x$. Compare your results with the true derivatives.

19.10 Develop an M-file function that computes first and second derivative estimates of order $O(h^2)$ based on the formulas in Figs. 19.3 through 19.5. The function's first line should be set up as

```
function [dydx, d2ydx2] = diffeq(x,y)
```

where `x` and `y` are input vectors of length `n` containing the values of the independent and dependent variables, respectively, and `dydx` and `dy2dx2` are output vectors of length `n` containing the first- and second-derivative estimates at each value of the independent variable. The function should generate a plot of `dydx` and `dy2dx2` versus `x`. Have your M-file return an error message if **(a)** the input vectors are not the same length, or **(b)** the values for the independent variable are not equally spaced. Test your program with the data from Prob. 19.11.

19.11 The following data was collected for the distance traveled versus time for a rocket:

t, s	0	25	50	75	100	125
y, km	0	32	58	78	92	100

Use numerical differentiation to estimate the rocket's velocity and acceleration at each time.

19.12 A jet fighter's position on an aircraft carrier's runway was timed during landing:

***t*, s**	0	0.52	1.04	1.75	2.37	3.25	3.83
***x*, m**	153	185	208	249	261	271	273

where x is the distance from the end of the carrier. Estimate **(a)** velocity (dx/dt) and **(b)** acceleration (dv/dt) using numerical differentiation.

19.13 Use the following data to find the velocity and acceleration at $t = 10$ seconds:

Time, *t*, s	0	2	4	6	8	10	12	14	16
Position, *x*, m	0	0.7	1.8	3.4	5.1	6.3	7.3	8.0	8.4

Use second-order correct **(a)** centered finite-difference, **(b)** forward finite-difference, and **(c)** backward finite-difference methods.

19.14 A plane is being tracked by radar, and data is taken every second in polar coordinates θ and r.

***t*, s**	200	202	204	206	208	210
***θ*, (rad)**	0.75	0.72	0.70	0.68	0.67	0.66
***r*, m**	5120	5370	5560	5800	6030	6240

At 206 seconds, use the centered finite-difference (second-order correct) to find the vector expressions for velocity $\vec{v}$ and acceleration $\vec{a}$. The velocity and acceleration given in polar coordinates are

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \quad \text{and} \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

19.15 Use regression to estimate the acceleration at each time for the following data with second-, third-, and fourth-order polynomials. Plot the results:

t	1	2	3.25	4.5	6	7	8	8.5	9.3	10
v	10	12	11	14	17	16	12	14	14	10

19.16 The normal distribution is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Use MATLAB to determine the inflection points of this function.

19.17 The following data was generated from the normal distribution:

x	−2	−1.5	−1	−0.5	0
***f*(*x*)**	0.05399	0.12952	0.24197	0.35207	0.39894

x	0.5	1	1.5	2
***f*(*x*)**	0.35207	0.24197	0.12952	0.05399

Use MATLAB to estimate the inflection points of this data.

19.18 Use the `diff(y)` command to develop a MATLAB M-file function to compute finite-difference approximations to the first and second derivative at each x value in the table below. Use finite-difference approximations that are second-order correct, $O(x^2)$:

x	0	1	2	3	4	5	6	7	8	9	10
y	1.4	2.1	3.3	4.8	6.8	6.6	8.6	7.5	8.9	10.9	10

19.19 The objective of this problem is to compare second-order accurate forward, backward, and centered finite-difference approximations of the first derivative of a function to the actual value of the derivative. This will be done for

$$f(x) = e^{-2x} - x$$

(a) Use calculus to determine the correct value of the derivative at $x = 2$.

(b) Develop an M-file function to evaluate the centered finite-difference approximations, starting with $x = 0.5$. Thus, for the first evaluation, the x values for the centered difference approximation will be $x = 2 \pm 0.5$ or $x = 1.5$ and 2.5. Then, decrease in increments of 0.1 down to a minimum value of $\Delta x = 0.01$.

(c) Repeat part **(b)** for the second-order forward and backward differences. (Note that these can be done at the same time that the centered difference is computed in the loop.)

(d) Plot the results of **(b)** and **(c)** versus x. Include the exact result on the plot for comparison.

19.20 You have to measure the flow rate of water through a small pipe. In order to do it, you place a bucket at the pipe's outlet and measure the volume in the bucket as a function of time as tabulated below. Estimate the flow rate at $t = 7$ s.

Time, s	0	1	5	8
Volume, cm³	0	1	8	16.4

19.21 The velocity v (m/s) of air flowing past a flat surface is measured at several distances y (m) away from the surface.